<u>Cor</u> Every fin. ext. of $\mathbb{Q}_p$ is a completion of some number field.

[I.e. local fields are exactly the completions of global fields (number fields, $\mathbb{F}_q(C)$)]

<u>Pf</u> $K/\mathbb{Q}_p$ finite; char $K = 0$ $\Rightarrow$
$K = \mathbb{Q}_p(\alpha)$, α root of some irr. monic poly in $\mathbb{Z}_p[X]$.

$\left.\begin{array}{l}\mathbb{Z} \text{ dense in } \mathbb{Z}_p \\ \text{Krasner's Lemma}\end{array}\right\} \Rightarrow$ there is a monic poly in $\mathbb{Z}[X]$ that defines the same field. ∎

§ Ramification & inertia

$L, |\cdot| \supset K, |\cdot|$ complete non-Archimedean, $[L:K] = d$.

Prop If L/K is separable, valuations are discrete then $\mathcal{O}_L \cong \mathcal{O}_K^d$ as $\mathcal{O}_K$-modules

both conditions are necessary, as well as completeness; e.g. not true for number field.

Pf $\mathcal{O}_L/\mathcal{O}_K$ integral $\Rightarrow tr_{L|K}(\mathcal{O}_L) \subseteq \mathcal{O}_K$.

$$B : L \times L \longrightarrow K \qquad \text{trace form}$$
$$x, y \longmapsto tr_{L|K}(xy)$$

L/K separable $\Leftarrow$ B non-degenerate bilinear form.

$e_1, \ldots, e_d$ basis for L/K.

$e_1^\vee, \ldots, e_d^\vee \in L$ dual basis ; $B(e_i, e_j^\vee) = \delta_{ij}$

Then

$$\mathcal{O}_L \subseteq \{x \in L \mid B(x,y) \in \mathcal{O}_K \mid \forall y \in \mathcal{O}_L\}$$

$$\subseteq \bigoplus_{i=1}^{d} \mathcal{O}_K \cdot e_i^\vee \quad \longleftarrow \text{fin.gen. free } \mathcal{O}_K\text{-module}$$

$\mathcal{O}_K$ PID $\Rightarrow$ $\mathcal{O}_L$ fin.gen.free$/\mathcal{O}_K$; rank d $\Rightarrow$ $\mathbb{Q}$.

<u>Rmk</u> $\delta_{L/K}^{-1} = \mathcal{O}_L^\vee = \{x \in L \mid tr(xy) \in \mathcal{O}_K \ \forall y \in \mathcal{O}_L\}$

$\mathcal{O}_L$-module, $\supseteq \mathcal{O}_L$.

Its inverse $\delta_{L/K} \subseteq \mathcal{O}_L$ is an ideal, called the <u>different</u> of L/K.

and $\Delta_{L/K} = \mathrm{Norm}_{L/K}\, \delta_{L/K}$ $(\subseteq \mathcal{O}_K)$

the <u>discriminant</u> of L/K [$\leftarrow$ measures the ramification of L/K)

(= discriminant of the quadratic form

$\mathrm{tr}_{L/K}(x^2) : \mathcal{O}_L \to \mathcal{O}_K.$)

Ex $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$

$L = \mathbb{Q}_p(\sqrt{p})$, $p \neq 2$, $\mathcal{O}_L = \mathbb{Z}_p[\sqrt{p}]$

$$tr : L \longrightarrow K$$
$$a + b\sqrt{p} \longmapsto 2a$$

$2ac + 2pbd$

$$\delta_{L|K} = \{ a + b\sqrt{p} \mid a, b \in \mathbb{Q}_p,\ tr((a+b\sqrt{p})(c+d\sqrt{p})) \text{ is in } \mathcal{O}_K \text{ for every } c, d \in \mathbb{Z}_p \}$$
$$= \{ a + b\sqrt{p} \mid a \in \mathbb{Z}_p,\ b \in \tfrac{1}{p}\mathbb{Z}_p \} = \tfrac{1}{\sqrt{p}}\mathcal{O}_L$$

$\delta_{L|K} = (\sqrt{p}) \subseteq \mathcal{O}_L$; $\Delta_{L|K} = \mathrm{Norm}_{L|K}(\sqrt{p})$

$= (\sqrt{p} \cdot \sqrt{p}) = (p) \subseteq \mathbb{Z}_p.$

different

discriminant

$\Delta_{L|K}$ = ideal generated by the discriminant of the quadratic form

$$tr(x^2) = tr((a+b\sqrt{p})^2) =$$
$$= 2a^2 + 2p\,b^2$$

which is $\det\begin{pmatrix} 2 & 0 \\ 0 & 2p \end{pmatrix} = (4p) = (p)$.

<u>Ex</u> Similarly for $p=2$,

$\delta_{L|K} = (2\sqrt{2})$, $\Delta_{L|K} = (8)$

From now on

K local non-Archimedean,
$\mathcal{O}_K$, $|\cdot|$ discrete v_K, π_K, res. field k_K (finite)

L/K finite, $[L:K]=d$, res. field k_L
$\mathcal{O}_L$, $|\cdot|_L$, v_L, π_K, $\left(=|N_{L/K}(\cdot)|^{1/d}\right.$ discrete as well$)$

<u>Def</u> $f = f_{L|K} = [k_L : k_K]$ <u>residue degree</u> (or inertial degree).

$e = e_{L|K} = [v_L(L^\times) : v_L(K^\times)]$ <u>ramification degree</u>

equivalently $(\pi_L^e) = (\pi_K) \subseteq \mathcal{O}_L$.

<u>Rmk</u> Two opposite conventions: (both used)

(i) normalise $v_K : K^\times \twoheadrightarrow \mathbb{Z}$,
$v_L : L^\times \twoheadrightarrow \mathbb{Z}$

so that $v_L\big|_{K^\times} = e v_K$.

(ii) let $v_L : K^\times \twoheadrightarrow \frac{1}{e}\mathbb{Z}$ (not normalised),

so that $v_L\big|_{K^\times} = v_K$ ← we'll do this

Prop $e_{L|K} f_{L|K} = [L:K]$

Pf $d = [L:K]$. As $\mathcal{O}_K$-modules,

$$\mathcal{O}_L \cong \mathcal{O}_K^d$$

$$\mathcal{O}_L / \pi_K \mathcal{O}_L \cong \left(\mathcal{O}_K / \pi \mathcal{O}_K\right)^d \cong k_K^d$$

but also

$$\mathcal{O}_L \supseteq \pi_L \mathcal{O}_L \supseteq \pi_L^2 \mathcal{O}_L \supseteq \cdots \supseteq \pi_L^e \mathcal{O}_L = \pi_K \mathcal{O}_L$$

all successive quotients $\cong \mathcal{O}_L / \pi \mathcal{O}_L \cong k_L$

e steps $\Rightarrow \left|\mathcal{O}_L / \pi_K \mathcal{O}_L\right| = \left|k_L^e\right| = \left|k_K^{f e}\right|$ ∎

<u>Rmk</u> Given $M/L/K$

$$f_{M/K} = f_{M/L} f_{L/K} \quad \text{by tower law for } k_M/k_L/k_K$$

and so

$$e_{M/K} = e_{M/L} e_{L/K} \quad \text{by tower law for } M/L/K.$$

<u>Def</u> L/K is <u>unramified</u> if $e=1$, $f=d$.

L/K is <u>totally ramified</u> if $e=d$, $f=1$.

Ex Quadratic extensions of $\mathbb{Q}_p$, $p \neq 2$.

$\eta \in \mathbb{Z}_p^\times$ reducing to $\bar{\eta} \in \mathbb{F}_p^\times \setminus \mathbb{F}_p^{\times 2}$

(quad. nonresidue mod p).

$\mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{\eta p})$ $\Rightarrow$ $e=2$, $f=1$.

have elt. $\sqrt{\ }$ of valuation $1/2$

$\mathbb{Q}_p(\sqrt{\eta})$ contains roots of $x^2-\eta$

$\Rightarrow$ residue field has roots of $x^2-\bar{\eta} \in \mathbb{F}_p[x]$

$\Rightarrow$ res. field $\supseteq \mathbb{F}_{p^2}$ $\Rightarrow$ $f=2$, $e=1$.

<u>Exc</u> $p=2 \to \mathbb{Q}_2(\zeta_3) = \mathbb{Q}_2(\sqrt{-3}) = \mathbb{Q}_2(\sqrt{5})$

is quadratic unramified

The other 6 are all (totally) ramified.

§ <u>Unramified extensions</u>

Essentially unramified extension correspond to extensions of the residue fields :

Thm (i) If L/K unramified, then $\mathcal{O}_L = \mathcal{O}_K[\alpha]$, $L = K(\alpha)$ for any $\alpha \in \mathcal{O}_L$ with $k_L = k_K(\bar{\alpha})$.

(ii) L/K unramified, L'/K any finite ext. Then

$$\mathrm{Hom}_K(L, L') \longrightarrow \mathrm{Hom}_{k_K}(k_L, k_{L'})$$

is a bijection

(iii) Suppose ℓ/k_K finite. Then there is a unique (up to $\cong$) unr. ext. L/K with $k_L \cong \ell$ over k_K. It is Galois and $\mathrm{Gal}(L/K) \overset{\sigma \mapsto \bar{\sigma}}{\cong} \mathrm{Gal}(k_L/k_K)$

<u>Pf</u> (i) $\mathcal{O}_L \cong \mathcal{O}_K^d$, $d = [L:K]$, $\pi_L = \pi_K$ (L/K unr.)

$k_L = k_K(\bar{\alpha}) \Rightarrow 1, \bar{\alpha}, \dots, \bar{\alpha}^{d-1}$ basis of k_L/k_K

$\Rightarrow 1, \alpha, \dots, \alpha^{d-1}$ generate $\mathcal{O}_L / \pi \mathcal{O}_L$

$\mathcal{O}_L$ local

$\Rightarrow 1, \alpha, \dots, \alpha^{d-1}$ generate $\mathcal{O}_L$

as an $\mathcal{O}_K$-module by Nakayama's lemma.

(ii) Write $L = K(\alpha)$ as in (i),

α root of some irr. monic $f(x) \in \mathcal{O}_K[x]$,

$\overline{f(x)}$ defines k_L over k_K, in particular

it is irreducible $/k_K \Rightarrow$ has distinct roots in k_L.

$$\mathrm{Hom}_K(L, L') \overset{1-1}{=} \{\text{roots of } f(x) \text{ in } L'\} \quad \underset{\xleftarrow{\text{Hensel}}}{\xrightarrow[\text{reduce}]{1:1}}$$

$$\{\text{roots of } \overline{f(x)} \text{ in } k_{L'}\}$$

$$= \mathrm{Hom}_{k_K}(k_L, k_{L'})$$

(iii) let $[\ell : k_K] = d$,

Write $\ell = k_K(\beta)$ (primitive elt. thm)

β root of a monic poly $F(x) \in k_K[x]$,

$\deg F = d$.

Lift $F(x)$ to a monic $f(x) \in \mathcal{O}_K[x]$, let

$L = K[x]/f(x)$. Then

$$\left.\begin{array}{l} f_{L|K} \geq \deg F = d \\ [L:K] \leq d \end{array}\right\} \Rightarrow \begin{array}{l} f_{L|K} = d \\ e_{L|K} = 1 \end{array}$$

so $L|K$ unr. and $k_L \cong \ell$.

Uniqueness: (ii) $\Rightarrow$ any two such are isomorphic over K

Galois: $|\mathrm{Aut}(L/K)| \overset{(ii)}{=} |\mathrm{Aut}(k_L/k_K)| = $

exts of finite fields are Galois

$= [k_L : k_K] = [L:K]$

$\Rightarrow L/K$ Galois

<u>Cor</u> $L|K$ finite, $e = e_{L|K}$, $f = f_{L|K}$.

There is a unique intermediate field M,

$$K \xhookrightarrow[\text{of degree } f]{\text{unramified}} M \xhookrightarrow[\text{of degree } e]{\text{totally ramified}} L.$$

This M is called the <u>maximal unramified extension</u> of K in L (= compositum of all unr. exts. of K in L).

$$
\begin{array}{ccccccc}
 & & L & & \mathcal{O}_L & (\pi_L) & k_L \cong \mathbb{F}_{q^f} \\
 & \text{(ii)} & \Big| & \text{tot. ram.} & \Big| & & \\
L' & = & M & & \mathcal{O}_M & (\pi_K) & k_M = \mathbb{F}_{q^f} \\
 & & \Big| & \text{unr.} & \Big| & & \\
 & \text{(iii)} & K & & \mathcal{O}_K & (\pi_K) & k_K = \mathbb{F}_q
\end{array}
$$

<u>Pf</u> Thm (iii) $\Rightarrow$ $\exists!$ unr. ext. $M|K$ with $k_M \cong k_L$ over k_K.

(ii) $\Rightarrow$ $M \hookrightarrow L$.

Now compare degrees. ∎

Recall: A finite field $\mathbb{F}_q$ has a unique (up to $\cong$) extension $\mathbb{F}_{q^n}$ of degree n;

$$\mathbb{F}_{q^n} = \mathbb{F}_q\left((q^n-1)^{\text{th}} \text{ roots of unity}\right)$$

= splitting field of $X^{q^n-1}-1$ (or $X^{q^n}-X$) over $\mathbb{F}_q$.

So every local field K has a unique <u>unramified</u> extension of degree n

$$L = K\left((q^n-1)\text{th roots of unity}\right)$$

$$= \text{splitting field of } X^{q^n-1}-1 \ \left(\text{or } X^{q^n}-X\right)$$

over K

where $q = \# k_K$.

$\underline{\text{Ex}}$ $K = \mathbb{F}_p((t))$

Its unramified extension of degree n is

$$L = \mathbb{F}_{p^n}((t))$$

$\underline{\text{Ex}}$ $\mathbb{Q}_2$

$n=1$	$\mathbb{Q}_2$	$\mathbb{F}_2$
$n=2$	$\mathbb{Q}_2(\zeta_3)$	$\mathbb{F}_4$
$n=3$	$\mathbb{Q}_2(\zeta_7)$	$\mathbb{F}_8$
$n=4$	$\mathbb{Q}_2(\zeta_{15}) = \mathbb{Q}_2(\zeta_5)$	$\mathbb{F}_{16}$

§ Totally ramified extensions

Recall: $g \in \mathcal{O}_K[X]$ is <u>Eisenstein</u> if

$$g(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1 X + a_0$$

with all $a_i \equiv 0 \mod \pi_K$ and $a_0 \not\equiv 0 \mod \pi_K^2$

(i.e. $v_K(a_i) > 0 \ \forall i$, $v_K(a_0) = 1$ if $v_K : K^\times \twoheadrightarrow \mathbb{Z}$).

Eisenstein's criterion $\Rightarrow$ such polynomials are irreducible.

Thm Suppose L/K is totally ramified of degree e, $\pi = \pi_L$ uniformiser, and $v_L : L^\times \twoheadrightarrow \mathbb{Z}$ normalised valuation. Then

(i) π satisfies an Eisenstein polynomial of degree e over $\mathcal{O}_K$.

(ii) $\mathcal{O}_L = \mathcal{O}_K[\pi]$ $\left(= \mathcal{O}_K + \pi\mathcal{O}_K + \dots + \pi^{e-1}\mathcal{O}_K\right)$

Conversely, if $g \in \mathcal{O}_K(x)$ is Eisenstein then

$L = K[X]/g(x)$ is totally ramified over K,

and $v_L(\text{root of } g) = 1$.

<u>Proof</u>: (i) Consider the min.poly of π over K,

$$\pi^n + a_{n-1}\pi^{n-1} + \dots + a_0 = 0$$

irreducible over K and $n \leq e = [L:K]$.

π is integral over $\mathcal{O}_K \Rightarrow a_i \in \mathcal{O}_K$ for $i = 0, \ldots, n-1$.

Now look at the valuations of terms : the sum is 0, so two terms (at least) have the smallest valuation

$$v_L(a_i \pi^i) = i + e\, v_K(a_i) \equiv i \pmod{e}.$$

Hence $n = e$, $v_L(a_0) = v_L(\pi^n) = n = e$,
in other words $v_K(a_0) = 1$; and $v_K(a_i) > 0$
because $v_K(a_0) = v_K(a_n)$ are smallest

Thus $g(X)$ is Eisenstein, irreducible,

and $L = K(\pi)$, $\mathcal{O}_L = \mathcal{O}_K(\pi)$